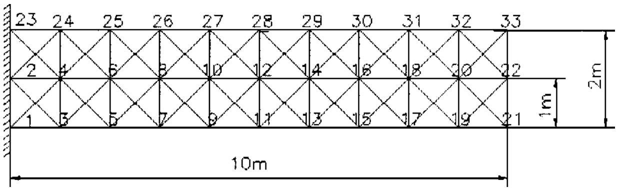

Fig. 3 Modified structure with 33 nodes and 90 members.

condensation.[7] That is because Guyan's static condensation does not contain the effects of mass from added members in the modified structure, whereas these effects are considered in the proposed method. From Eqs. (10) and (16–18), it can be seen that little computational effort is added in the proposed method. From the results, it can be also seen that the proposed method can also give high-quality accuracy for the large topological modifications.

Conclusions

A hybrid method is presented for the efficient calculation of eigenpairs of topological modifications in dynamic problems. The effects of mass from the added member are considered in this method. From Eqs. (10) and (16–18), it can be seen that the little computational effort is added in the proposed method. The results show that the proposed method is efficient for eigenproblems of topological modifications. For large topological modifications, the presented method can also give high accuracy.

Acknowledgments

This work is supported by the National Natural Science Foundation of the People's Republic of China and the Natural Science Foundation of the South China University of Technology.

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Toward More Effective Genetic Algorithms for the Optimization of Piezoelectric Actuator Locations

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Introduction

BECAUSE of the well-known advantages of adaptive structures over traditional structures, significant research is being conducted on these structures at present (for example, Ref. 1). One key requirement toward the success of adaptive structures would be in choosing the optimal location of actuators. A large number of publications[2–5] have addressed this problem. In our previous paper[5] various methods addressing this issue were mentioned. Among various approaches being used, genetic algorithms (GA) were found to be popular. Some advantages and disadvantages of GAs were presented in our previous paper.[5] In previous studies the present authors successfully applied two versions of GAs (termed GA version 1 and GA version 2). Both of the versions were adapted from Carroll's FORTRAN Genetic Algorithm Driver, to solve two kinds of difficult, computationally intensive, combinatorial, and continuous large-scale optimization problems [code available online at http://www.aic.nrl.navy.mil:80/galist/src/#fortran (cited 3 May 2002)]. These problems include finding both an optimal placement and optimal voltages of 30 piezoelectric actuators, from 193 candidate locations, with more than 1.28×10^{35} possible solutions to obtain the best correction to the surface thermal distortions of a thin hexagonal spherical primary mirror (Fig. 1a in Ref. 5) of an astronomical telescope. The thermal distortions were caused by four different types of spatial temperature distributions. The two types of optimization problems were as follows: 1) to find the optimal locations and optimal voltages suitable for each type of thermal loads individually and 2) to determine just one set of actuator locations that would reduce the distortion caused by all four types of thermal loads. The latter problem is a more challenging, multicriterion optimization problem. A laminated triangular shell element[6] was used to model the mirror. The main conclusions from our previous studies are as follows: 1) the design search space is highly multimodal; 2) both GA version 1 and GA version 2 are effective for the optimization of piezoelectric actuator locations; 3) GA version 2 has more flexibility than GA version 1; 4) GA version 2 can get modestly better results than DeLorenzo algorithm for both optimization problems for the case of 30 piezoelectric actuators; 5) the convergence to a solution can occur without reaching an optimal or near-optimal solution; 6) more than one suboptimal solution to each problem was found; 7) optimal location obtained for one type of thermal loads may perform poorly for other types of thermal loads; and 8) GAs can determine one set of actuator locations, which is good for all four of the types of thermal loads considered for these studies. The needed voltages will be different for different thermal loads.

In the present Note an improved GA, termed GA version 3 and adapted from the GA version 2, is employed to resolve the two

Presented as Paper 2001-1627 at the AIAA/ASME/ASCE/AHS/AHC 42nd Structures, Structural Dynamics, and Materials Conference, Seattle, WA, 16–19 April 2001; received 16 July 2001; revision received 21 January 2002; accepted for publication 15 February 2002.

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problems studied in Ref. 5 and even larger problems. Two key differences between the GA version 3 and the earlier GA version 2 are 1) the application of random-mutation hill climbing to elitists and 2) the application of mutation to microgenetic algorithms. The results of using same parameter settings of the GA version 3 to even larger problems, that is, choosing a set of 121 piezoelectric actuator locations from 193 candidate locations with more than 1.38×10^{54} possible solutions, are also reported.

Genetic Algorithms

GAs,[7–13] inspired by natural evolution, have drawn considerable attention during the past two decades as a result of their ability to solve large complex optimization problems that might be difficult to solve using conventional gradient-based optimization techniques. GAs are robust stochastic global search techniques based on the mechanics of natural selection and genetics. These algorithms evolve a population of chromosomes using selection and genetic operations such as crossover, mutation, and so on from one generation to another, hopefully to get a better solution. The selection operation is based on the Darwinian principle of the survival of the fittest. Genetic algorithms use only one very general assumption: namely, better individuals can reproduce better offspring more probably than the worst individuals. The concept's simplicity, flexibility, and robust performance makes GAs one of the most exciting fields in evolutionary computation.

In the field of shape, acoustic and vibration, buckling, and aeroelastic control of smart structures, the effectiveness of the control system strongly depends on the actuator locations. In this Note the improved GA (termed version 3), developed by the present authors from an earlier GA, the GA version 2, is employed to solve an important problem in the design of smart structures, namely, the selection of actuator locations. Two key differences between GA version 3 and GA version 2 are as follows: 1) the application of random-mutation hill climbing to elitists and 2) the application of mutation to micro-genetic algorithms.

First, random-mutation hill climbing was applied to elitists, the best individuals. This came from our initial intuition that, if we do not know which individual can reproduce a better offspring, choosing the best one generally gets the highest probability. (We need to recall from Ref. 7, p. 201, "in their purest form, genetic algorithms are blind search procedures," but in practical form GAs are directed search techniques and not completely blind.) Random-mutation hill climbing outperformed the steepest-ascent hill climbing and next-ascent hill climbing (Ref. 9, p. 129).

Second, mutation was applied to microgenetic algorithms. The microgenetic algorithms here are the same as the genetic algorithms in the commonly used sense except that they include restart function in the outer loop and usually use a small population size in order to get the effect of faster convergence rate than the large population size usually used in genetic algorithms. In the literature, the mutation rate was set at 0.0 in most cases whenever the microgenetic algorithms were used. This is probably because of the traditional negative view of mutation, namely that the use of mutation slows convergence. Recent advances in genetic algorithms show that the mutation operation also has some positive effects: speeding convergence as well as providing the diversity of population, thereby avoiding a premature convergence to a local optimum.

Problem Definition

In the design of the next generation of astronomical telescopes, one of the most stringent requirements will be the maintenance of high surface accuracy of the primary mirror during their operation. A promising method is to use a certain number of piezoelectric actuators bonded onto the rear surface of the primary mirror to correct its distortions without imposing a significant weight penalty. The problem is how to find the optimal location of piezoelectric actuators to maximize their effectiveness. Our problem is as follows:

With n, the number of piezoelectric actuators available, determine from a total of 193 candidate locations an optimal placement and corresponding optimal voltage for each actuator to obtain the best correction to the surface thermal distortions of a thin hexagonal spherical primary mirror subjected to four different types of thermal loads (Fig. 1a in Ref. 5). There are two kinds of optimization problems: one is to find a set of locations and corresponding voltages that get the best correction to the surface thermal distortions under each of the four types of thermal loads; the other is to find one set of locations and corresponding voltages that provide the best possible correction to the surface thermal distortions caused by all of the four types of thermal loads. For the second problem, although the actuator locations are the same for all the four thermal distortions, the corresponding voltages might not be. The second problem is a multicriterion problem and obviously is a more challenging problem. The total number of different candidate sets are

$$^{193}C_n = \binom{193}{n} = \frac{193!}{n!(193-n)!}$$

The geometry and material properties of the mirror and piezoelectric actuators are given in Table 4 in Ref. 3. The four types of temperature distributions at the lower surface of the mirror are given in Table 5 of Ref. 3.

Finite Element Modeling

A laminated triangular shell element[6] is used to model the mirror. The element is a combination of the discrete Kirchhoff theory plate bending element and a membrane element derived from the linear strain triangular element with a total of 18 degrees of freedom (three translations and three rotations per node). The piezoelectric strips are assumed to be perfectly bonded on the lower surface of the mirror and are modeled as a separate layer. The finite element model consists of 864 flat shell elements, 469 grid points (Fig. 1b in Ref. 5). The mirror segment is assumed to be simply-supported at the six vertices 1, 13, 223, 247, 457, and 469.

Control Algorithms

The surface thermal distortions or the transverse displacements w of the mirror segment are corrected by applying the voltage across the thickness of the strip, which induces a distributed strain in the strip and hence in the mirror. In this study the thermal deformation w caused by any one type of thermal loads is computed by the finite element analysis. The finite element formulation suggested in Ref. 6 is capable of analyzing panels under thermal loads. The corresponding formulations for the correction u_i and rms error E are available in Ref. 5.

For each set of locations, we can get the optimal voltages to minimize rms error. Different settings of actuator location have different optimal voltages and corresponding minimum rms error. Thus, the first optimization problem is to find a set of locations and corresponding voltages that minimizes the minimum rms error for one type of distortion that is of the form

$$E = \operatorname*{Min}_{L} \operatorname*{Min}_{V} E(T, L, V)$$

The second optimization problem is to find a set of locations and corresponding voltages that minimize the maximum of the minimum rms error for all of the four different distortions that are of the form

$$E = \operatorname*{Min}_{L} \operatorname*{Max}_{T} \operatorname*{Min}_{V} E(T, L, V)$$

Obviously, the second problem is a more realistic, but computationally challenging, problem.

Results and Discussion

In this section the results obtained by using the GA version 3 developed by the present authors from the GA version 2[5] to solve the preceding two kinds of optimization problems are presented. The following parameters were used.

Version 1: population size 5, crossover rate = 0.5, mutation rate = 0.0, restart control parameter diffrac = 0.06.

Version 2: run1–initial population size 10, population size 5, scale = 0.5, random = 0, crossover rate = 0.5, mutation rate = 0.0, restart control parameter diffrac = 0.06.

Version 2: run2–initial population size 10, population size 5, scale = 0.5, random = 0, crossover rate = 0.5, mutation rate = 0.0, restart control parameter diffrac = 0.0.

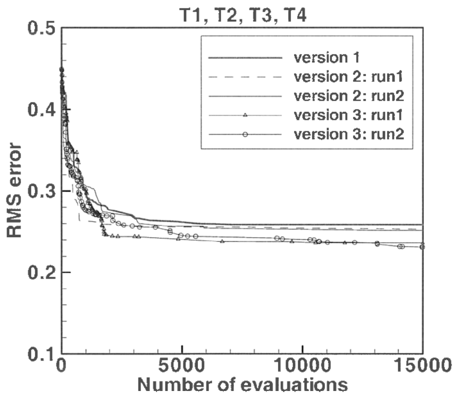


Fig. 1 Performance of the GAs: maximum of the four minimum rms errors vs the number of evaluations for the second optimization problem (30 actuators).

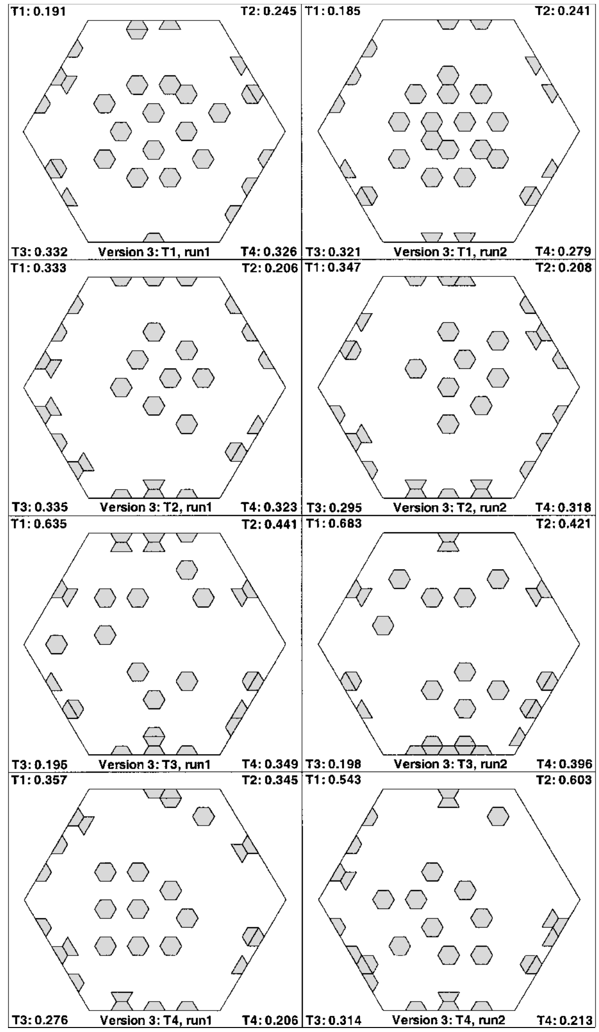
Version 3: run1–initial population size 10, population size 5, scale = 0.5, random = 0, crossover rate = 0.5, mutation rate = 0.01, No_of_max_generation_inner_loop = 15, No_of_best_mutation_bits = 2.

Version 3: run2–initial population size 10, population size 5, scale = 0.5, random = 0, crossover rate = 0.5, mutation rate = 0.01, No_of_max_generation_inner_loop = 10, No_of_best_mutation_bits = 2.

Note the following: The new parameter "No_of_best_mutation_bits," which represents mutation bits for elitists in each generation, was introduced in version 3. The parameter "scale" is used to adjust the selective pressure. The parameter "random" is used to control whether the initial population size and population size are randomly generated or not. When the parameter "random" equals 0, the initial population size and population size equal the preset values respectively; otherwise, they equal the numbers generated randomly. The parameter "diffrac" is used to check the convergence of population. When this value becomes less than the preset value, the new population are randomly generated.

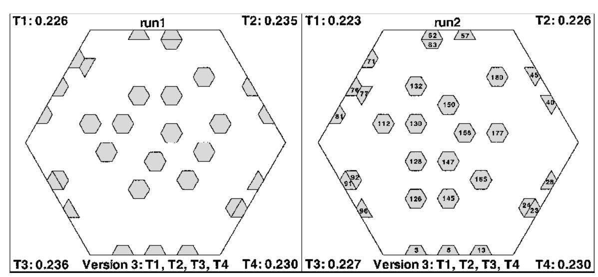
In this study the number of evaluations using genetic algorithms is limited to 15,000. For 30 piezoelectric actuators Fig. 1 shows the performance of the GAs with respect to the number of evaluations for the second optimization problem. The optimal actuator locations obtained by the two runs of the GA version 3 for the two kinds of optimization problems are presented in Figs. 2 and 3, respectively. Figure 2 presents the results for the first kind of optimization problems for each of the four types of thermal loads; Fig. 3 presents the results for the second kind of optimization problem for all four types of thermal loads. The number in each corner in Figs. 2 and 3 represents the rms error corresponding to that type of thermal distortion. The set of actuator locations, shown in the top-left corner of Fig. 2, was obtained to minimize the error caused by thermal load T1. The rms error under thermal load T1 becomes 0.191; the error under T2, T3, and T4 becomes 0.245, 0.332, and 0.326, respectively. Hence it is seen that this set of actuator locations, although best for T1, might perform poorly when used for other types of thermal loads.

Regarding the performance of GAs (Fig. 1), GA version 3 not only approached near-optimal solution very fast but also found the best solution. For example, when used for the second kind of optimization problem, the multicriterion optimization problem (Fig. 1), GA version 3 run1 approached a near-optimal solution, with a distortion rms error value of 0.251, in fewer than 2000 evaluations and GA version 3 run2 obtained the same rms error in fewer than 5000 evaluations. Moreover, the latter obtained the best solution, with a distortion rms error value of 0.230 at the end of 15,000 evaluations, but GA version 2 run2 did not reach a near-optimal solution, with a distortion rms error of 0.251 until more than 13,000 evaluations. The GA version 2 run1 and GA version 1 did not even reach any near-optimal solution with the level of distortion rms error of 0.251, at the end of stipulated 15,000 evaluations. The research shows that the GA version 3 is more effective than the preceding two versions in solving optimization problem of determining actuator locations for thermal distortion control.



a) b)

Fig. 2 Optimal location obtained by a) GA version 3, run1, and b) GA version 3, run2 (30 actuators) for the first optimization problem for each of the four thermal loads.



a) b)

Fig. 3 Optimal location for the second optimization problem obtained by a) GA version 3, run1, and b) GA version 3, run2 (30 actuators).

Figures 4–7 present the results for the two kinds of optimization problems for the case of 121 piezoelectric actuators. This case was not presented in Ref. 5. The performance of GA version 3 for the second kind of optimization problems is shown in Fig. 4. We can see from these figures that the GA version 3 still reaches near-optimal solutions very fast even though the search space of 1.38×10^{54} different sets of actuator locations for this case is much larger than that for the case of 30 piezoelectric actuators. Figure 5 presents the actuator location for the first kind of optimization problem and corresponding rms errors obtained using the DeLorenzo's algorithm. Figures 6 and 7 present the actuator location for the two kinds of optimization problems and corresponding rms errors obtained by the two runs of the GA version 3. For the case of 121 actuators, we can

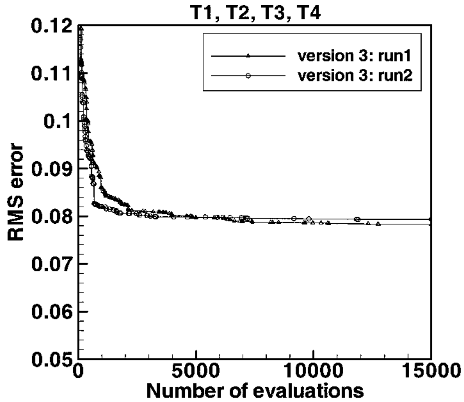


Fig. 4 Performance of the GAs: maximum of the four minimum rms errors vs the number of evaluations for the second optimization problem (121 actuators).

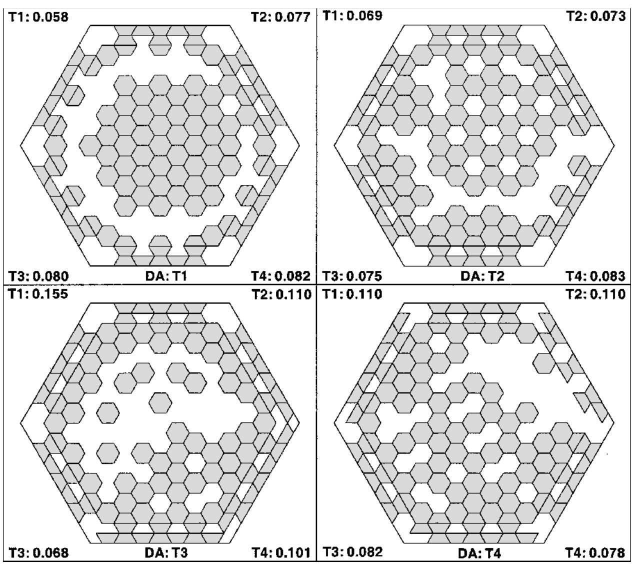
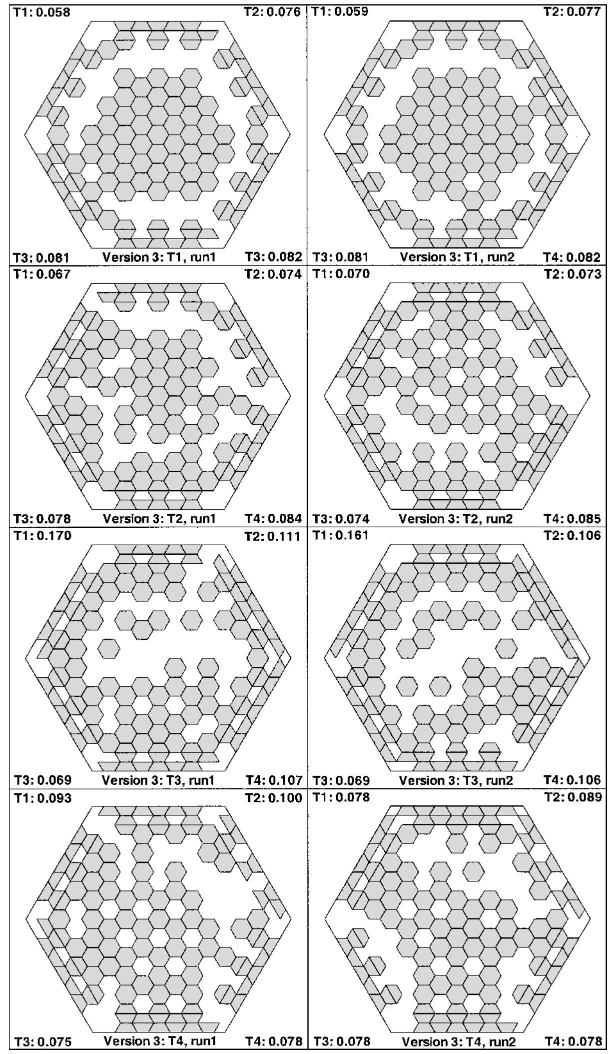
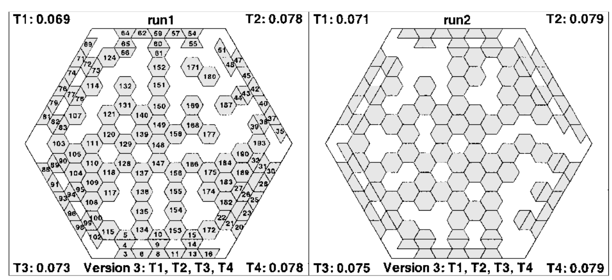


Fig. 5 Optimal location obtained by the DeLorenzo's algorithm (121 actuators) for the first optimization problem for each of the four thermal loads.



a) b)

Fig. 6 Optimal location obtained by a) GA version 3, run1, and b) GA version 3, run2 (121 actuators) for the first optimization problems for each of four thermal loads.



a) b)

Fig. 7 Optimal location for the second optimization problem obtained by a) GA version 3, run1, and b) GA version 3, run2 (121 actuators).

see in Figs. 5 and 6 that the rms errors for the first kind of optimization problems for each of the four different types of thermal loads obtained by the two runs of GA version 3 are basically the same as those obtained by the DeLorenzo's algorithm, but different sets of actuator locations are found. The performance of a set of actuator locations determined to be best for a given type of thermal loads does not deteriorate as much for other loads as it did in the case of 30 actuators. This implies that if one can employ a larger number of actuators the performance will be better for a large variation in the thermal loads. A large number of actuators will thus provide a more robust set of locations. Moreover, the magnitude of the rms error reduces as one employs a larger number of actuators. We can see that the rms error in the worst case T4 for the set of actuator locations in Fig. 7a for the second kind of optimization problem is as good as that in the best case T4 in Fig. 5 or 6 for the first kind of optimization problem. This demonstrates that the search space is highly multimodal and that the GA version 3 is very powerful to search the solution as good as possible for the multicriterion optimization problem.

The optimal voltages[14] corresponding to the optimization location for each type of thermal loads might be still too high to generate in space. Some promising methods can be used to lower the control voltages. For example, one can select the piezoelectric materials with higher strain constant as actuators or optimize the actuator location and corresponding voltages by applying constraints to electric voltages such as given the maximum voltages that can be provided. The latter is another different large and computationally intensive optimization problem.

Conclusions

In this study an improved version of GAs (version 3) developed from the GA version 2 by applying random-mutation hill climbing to elitists and applying mutation to microgenetic algorithms was used to solve two kinds of combinatorial and continuous large-scale optimization problems for two cases: selecting 30 and 121 actuator locations from 193 candidate locations in the design of a thin hexagonal spherical primary mirror to be used in the next generation of astronomical telescopes. One type of optimization problem is to find a set of locations and the corresponding voltages that give us the best correction to the surface thermal distortions of the primary mirror under a given type of thermal loads; the other is to find one set of locations and corresponding voltages that provide the best correction to the surface thermal distortions caused by all of the four different kinds of thermal loads. The two types of problems are difficult and computationally intensive. The second type is a more challenging, multicriterion optimization problem. The search space for these problems is highly multimodal, and conventional point-by-point optimization techniques usually get stuck at the local optimum, but population-based GAs are very good at searching such space and more likely get better results than the traditional techniques. The search space of 1.38×10^{54} different sets of actuator locations for the case of 121 piezoelectric actuators is much larger than that for the case of 30 piezoelectric actuators, but the GA version 3 still converged very fast and finally found a very good set of actuator locations that can be used to reduce all of the four kinds of thermal distortions. The results show that the two modifications employed in this study significantly improve the performance of GAs. The current version of GAs are more effective than the preceding versions in solving optimization problems of determining actuator locations for thermal distortion control.

The problems for this study are computationally intensive. Getting one solution using the GAs with the limit of 15,000 evaluations for the second optimization problem took more than a week. GAs are very general and robust optimization methods that can be applied to virtually any optimization problem. Parallel GAs can significantly reduce the time and be more natural to imitate the evolution of the nature, and so our current research is to develop parallel GAs.

Acknowledgments

The work was performed under a subcontract from the University of Texas at Arlington; themselves working under a grant from the National Science Foundation; and under a grant from the Carilion Biomedical Institute. We would like to thank the College of Engineering for providing computational resources for this research.

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Stability of Tapered Columns Under End-Concentrated and Variably Distributed Follower Forces

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Introduction

ELASTIC systems subjected to concentrated and distributed types of nonconservative follower forces are always encountered in engineering practices, such as in stability analysis of rockets, missiles, slender space structures, pipes conveying fluid, automobile disk, and drum brakes. A literature review indicates that the nonconservative stability of nonuniform structures was usually solved by the Galerkin method,[1] the finite difference method,[2] the finite element method (FEM),[3] and the transfer matrix method.[4] It is difficult to find the exact analytical solutions for stability of tapered structures with arbitrary boundary conditions. In this Note a successful attempt is made to present an efficient analytical method for the nonconservative stability of tapered columns. The closed-form solutions for the stability of two types of nonuniform columns subjected to an end concentrated and variably distributed follower forces are derived for the first time. The advantage of the proposed method is that the resulting characteristic equation for stability of a nonuniform column with any kind of two-end support configuration can be conveniently determined from a second-order determinant. As a consequence, the decrease in the determinant order, as compared with previously developed procedures, leads to significant savings in the computational effort.

Theory

A column with variable cross section under the combined action of an end concentrated follower force and variably distributed follower forces along the column is shown in Fig. 1. Considering the element shown in Fig. 2 and according to the d'Alembert principle, all of the forces acting on the element should satisfy the equilibrium conditions. From $\sum F_x = 0$, $\sum F_y = 0$, and using the method of separation of variables, one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[K(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}\right] + N(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} - \bar{m}(x)\omega^2 X(x) = 0 \quad (1)$$

where $X(x)$ is the mode shape function, ω is the circular natural frequency, $\bar{m}(x)$ is the mass per unit length of the column, $K(x)$ is the flexural stiffness, and $N(x)$ is the axial force.

Obviously, the solution of Eq. (1) is dependent on the expression of $K(x)$, $N(x)$, and $\bar{m}(x)$. As suggested by Li et al.[5,6] and Li,[7] the functions for describing the variations of $K(x)$, $N(x)$, and $\bar{m}(x)$, for many cases of structural members are power functions and exponential functions, which are considered in this Note.

Received 24 September 2001; revision received 27 February 2002; accepted for publication 2 March 2002.

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